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*Rapid Note*

# Dynamics of selfavoiding tethered membranes. II. Inclusion of hydrodynamic interaction (Zimm model)

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**Abstract.** The dynamical scaling properties of selfavoiding polymerized membranes with internal dimension $D$ embedded into $d$ dimensions are studied including hydrodynamical interactions. It is shown that the theory is renormalizable to all orders in perturbation theory and that the dynamical scaling exponent $z$ is given by $z = d$. The crossover to the region, where the membrane is crumpled swollen but the hydrodynamic interaction irrelevant is discussed. The results apply as well to polymers ($D = 1$) as to membranes ($D = 2$).

**PACS.** 05.70.Fh Phase transitions: general aspects – 11.10.Gh Renormalization – 11.25.-w Theory of fundamental strings

The statistical properties of polymerized flexible membranes, generalizing polymers, have found large interest during the last years. Due to the selfavoidance, they are either found in a flat or crumpled swollen phase [1–7]. An analytical approach was initiated in [8,9], where calculations of the static scaling exponent $\nu$ were performed at 1-loop order. Its consistency to all orders in perturbation theory has been established in [10,11]. Recently, 2-loop calculations have been performed, which give reliable results for all imbedding dimensions [12,13].

In this letter we want to address the question of the dynamics of such membranes in the crumpled swollen phase embedded in some (viscous) solvent. (For a discussion of the flat phase, see [14].)

We first summarize the main results before discussing the technical procedure to derive them.

First of all, the Brownian motion of the particles, both of the solvent and of the fluid, have to be taken into account. It is responsible for the relaxation of the membrane. This can be studied *via* the auto-correlation function which has for large membrane size and large time the scaling form

$$\left\langle (r(x,t) - r(x,0))^2 \right\rangle \sim t^{2/z}. \tag{1}$$

Our goal is to determine $z$.

In the physical system, hydrodynamics may be important. Two cases can be distinguished: In the first case, the hydrodynamic is irrelevant and the exponent $z$ is given by

$$z = 2 + d_{\mathrm{f}}, \tag{2}$$

where $d_{\mathrm{f}}$ is the fractal dimension of the membrane. This result has been established to all orders in perturbation theory in [15]. In the second case, hydrodynamic interactions are relevant and we will show below that this modifies the exponent $z$ to

$$z = d, \tag{3}$$

where $d$ is the dimension of the embedding space, see below. This situation is plotted in Figure 1, where also the phase-separation line is given.

The exponent $z$ should be observable *via* dynamic light- or neutron-scattering methods. To our knowledge, no such experiment has been performed.

Theoretically, the dynamics for polymers has first been regarded in [16,17] using scaling arguments. For membranes, a similar analysis has been performed in [18].

Scaling can best be studied using renormalization group methods. For polymers, such a treatment has been carried out at 1-loop order in [16,17,19–24]. A proof of the renormalizability which ensures the correctness of the method has been given in [15] for the case of purely dissipative motion (Rouse model).

Let us now introduce the model with hydrodynamic interactions, show that it is renormalizable and calculate the scaling exponent $z$.

The static behavior of the membrane is given by the Edwards Hamiltonian

$$\mathcal{H}[r] = \frac{1}{2}\int_x \Big(\nabla r(x)\Big)^2 + b\int_x\!\!\int_y \delta^d\Big(r(x) - r(y)\Big). \tag{4}$$

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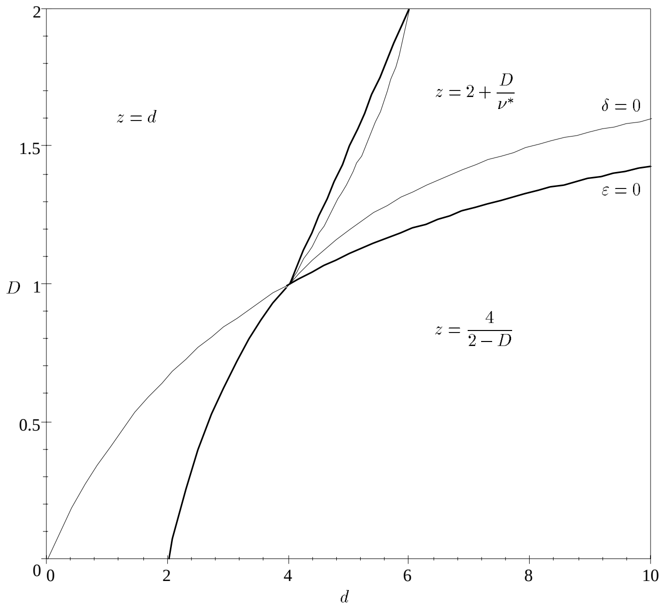


**Fig. 1.** The different phase-regions for a $D$-dimensional membrane embedded into $d$ dimensions including hydrodynamic interactions. The region with $\delta < 0$ and $\varepsilon < 0$ is the Gaussian phase. Selfavoidance and hydrodynamic interaction are irrelevant, *i.e.* $\nu^* = \frac{2-D}{2}$ and $z = \frac{4}{2-D}$. Hydrodynamics is relevant for small $d$ and becomes irrelevant if $d > d_{\mathrm{f}} + 2$, where $d_{\mathrm{f}} = D/\nu^*$ is the fractal dimension of the membrane. For $d > d_{\mathrm{f}} + 2$ and $\varepsilon > 0$, $z = 2 + d_{\mathrm{f}}$.

The embedding of the $D$-dimensional membrane in $d$-dimensional bulk space is described by the mapping $x \in \mathbb{R}^D \to r(x) \in \mathbb{R}^d$. $b$ is the coupling constant, associated to the self-avoidance.

Hydrodynamic interactions for polymers were first introduced by Zimm [25]. He wrote down the following Langevin equation, which we will also use for membranes:

$$\dot{r}(x,t) = \mathcal{D} \cdot \left( -\frac{\delta H}{\delta r} + \zeta \right)(x,t). \tag{5}$$

Here, $\cdot$ denotes the scalar product of the matrix operator $\mathcal{D}$ and the vector $\delta H/\delta r$, which is defined by

$$f \cdot g := \int_x f_\alpha(x) g_\alpha(x). \tag{6}$$

The hydrodynamic interaction is

$$\begin{aligned} &\mathcal{D}_{\alpha\beta}(x,y,r(x,t),r(y,t)) = \qquad (7)\\ &\lambda\delta_{\alpha\beta}\delta^D(x-y) + \lambda\eta \int_k \left( \frac{\delta_{\alpha\beta}}{k^2} - \frac{k_\alpha k_\beta}{k^4} \right) \mathrm{e}^{ik(r(x,t)-r(y,t))}. \end{aligned}$$

We will not repeat the derivation [25] of equation (7) here. Let us however note that one supposes that the hydrodynamic degrees of freedom are fast enough, so that their dynamics can be neglected and that screening effects are irrelevant. This might be wrong for membranes and in this case our results would only apply to membranes with large holes. For $\eta = 0$, (5) reduces to purely diffusive motion (Rouse model).

The noise correlation is

$$\overline{\zeta_\alpha(x,t)(\mathcal{D}\cdot\zeta)_\beta(y,t')} = 2\delta(t-t')\delta^D(x-y). \tag{8}$$

This ensures that the static behavior is correctly reproduced. Following Janssen [26], the corresponding field-theory is obtained by imposing the Langevin-equation through an auxiliary field $\tilde{r}$. Integration over the noise then yields the dynamic functional in Ito-discretization

$$J = \int_t \tilde{r}\cdot\dot{r} + \tilde{r}\cdot\mathcal{D}\cdot\frac{\delta H}{\delta r} - \tilde{r}\cdot\mathcal{D}\cdot\tilde{r}. \tag{9}$$

This model has to be renormalized. Analogously to [15] divergences only occur at small distances. They can be analyzed *via* a multilocal operator product expansion (MOPE). Renormalizability is ensured [15,10,11] if counter-terms for all possible marginal and relevant operators are included into the action. Due to causality, only operators with at least one response field $\tilde{r}$ are needed. These are the local operators $\tilde{r}\dot{r}$, $\tilde{r}(-\Delta)r$ and $\tilde{r}^2$. Other local operators like $\tilde{r}r^n$ are forbidden by translation-invariance in $r$-space.

By the same arguments one finds that there are no new marginal or relevant counter-terms proportional to 2- or 3-body interactions and 4-body-interactions are irrelevant. (Of course, long-range interactions are relevant, but they are not generated in perturbation theory.) We now want to show that the structure of the model is preserved, *i.e.* that it can be renormalized if we introduce renormalized quantities according to

$$\begin{aligned} \lambda &= Z_\lambda \lambda_{\mathrm{R}} \sqrt{\frac{Z}{\tilde{Z}}} \\ r &= \sqrt{Z} r_{\mathrm{R}} \\ \tilde{r} &= \sqrt{\tilde{Z}} \tilde{r}_{\mathrm{R}} \\ b &= b_{\mathrm{R}} Z_b Z^{d/2} \mu^\varepsilon \\ \eta &= \eta_{\mathrm{R}} Z_\eta Z_\lambda^{-1} Z^{d/2-1} \mu^\delta. \end{aligned} \tag{10}$$

The two regularization parameters $\varepsilon$ and $\delta$ are given by

$$\begin{aligned} \varepsilon &= 2D - \nu d \\ \delta &= 2 - \nu d \\ \nu &= \frac{2-D}{2}\cdot \end{aligned} \tag{11}$$

Perturbation theory is performed about the point ($\delta = 0$, $\varepsilon = 0$), *i.e.* ($D = 1$, $d = 4$). As the model is constructed such that the static limit is correctly reproduced, $Z$ and $Z_b$ are the renormalization-factors of the static theory [27].

Then, $\tilde{Z}$ is determined in order to render $\dot{r}$ finite.

The composition of $\tilde{r}$ with any other operator with the same time argument is always free of divergences, as the contraction of $\tilde{r}$ with any functional of $r$ and $\tilde{r}$ vanishes. We therefore conclude that the first term in the action, $\tilde{r}\dot{r}$ is correctly renormalized.

Then, $Z_\lambda$ and $Z_\eta$ are chosen to render the operator $\mathcal{D}$ finite. By the same arguments as above, the last term in the action, $\tilde{r}\cdot\mathcal{D}\tilde{r}$ is renormalized.

We still have to show that also the composite operator $\mathcal{D} \cdot \frac{\delta H}{\delta r}$ is finite. This is a consequence of the equation of motion obtained through variation of $J$ by $\tilde{r}(x,t)$:

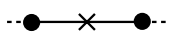
$$\dot{r}(x,t) + \left(\mathcal{D} \cdot \frac{\delta H}{\delta r}\right)(x,t) - 2\left(\tilde{r} \cdot \mathcal{D}\right)(x,t) = 0. \tag{12}$$

This relation is valid as an operator-identity. We have already renormalized $\dot{r}$, $\mathcal{D}$, $\frac{\delta H}{\delta r}$ and $\tilde{r}$. Equation (12) thus states that also the composite operator $\mathcal{D} \cdot \frac{\delta H}{\delta r}$ is finite. We can now conclude that all the terms in the action are finite.

There are three nontrivial relations which considerably simplify renormalization and which we are going to study now. First of all, due to the fact that in any interaction vertex the field $r$ appears either as difference $(r(x,t) - r(y,t))$ or as spatial derivative, no divergence proportional to $\tilde{r}\dot{r}$ appears. (For a detailed discussion see [15].) This means that

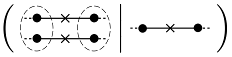
$$\tilde{Z} Z = 1. \tag{13}$$

In addition, there is no proper renormalization of the hydrodynamic interaction. Let us explain this point. Denote by



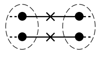
$$= \int_k \left(\frac{\delta_{\alpha\beta}}{k^2} - \frac{k_\alpha k_\beta}{k^4}\right) \mathrm{e}^{ik(r(x,t)-r(y,t))} \times f_\alpha(x,t) g_\beta(y,t) \tag{14}$$

any hydrodynamic interaction vertex. (The dotted line represents any polynomial in $r$ and $\tilde{r}$ or their derivatives.) Then singular configurations which give rise to a renormalization of the hydrodynamic interaction are those for which two interaction vertices are contracted to one single vertex. We claim that their multilocal operator product expansion (MOPE) does not contain a contribution proportional to the hydrodynamic interaction vertex:



$$= 0. \tag{15}$$

(The round dotted lines indicate points which are contracted.) This property is due to the analytic behavior of the long range (hydrodynamic) interaction. Dropping indices, the structure of such a contraction is



$$= \int_k \int_p k^{-2} p^{-2} \mathrm{e}^{i(k+p)(r(x,t)-r(y,t))} \times \mathrm{e}^{kp(C(\delta x,t)+C(\delta y,t))} + \text{subdominant}. \tag{16}$$

In order to obtain a long-range term a pole at $k+p=0$ is necessary. For $d > 2$ however, the expression is analytic. No long-range term is generated. This is easily generalized to any order in perturbation theory.

We now use the fluctuation-dissipation theorem

$$\Theta(t-t') \left\langle r_\alpha(x,t) \dot{r}_\beta(y,t') \right\rangle = \left\langle r_\alpha(x,t) \left(\tilde{r} \cdot \mathcal{D}\right)(y,t') \right\rangle, \tag{17}$$

which is derived along the same lines as in [26] and which we write down in Ito-discretization. (For other discretizations, additional terms appear on the r.h.s. which cancel the contraction of $\tilde{r}$ and $r$ with the same argument.) It is valid for bare and renormalized quantities. Inserting the definition of $\mathcal{D}_{\alpha\beta}$ we conclude that

$$Z_\eta = 1. \tag{18}$$

In the parameterization given above, $Z_\lambda$ is

$$Z_\lambda = 1 + \mathcal{O}(\eta_\mathrm{R}) \tag{19}$$

as it has to vanish for $\eta_\mathrm{R} = 0$. These relations are sufficient to completely solve for the anomalous exponents. We are interested in the IR-behavior. Suppose that the coupling related to the selfavoidance has flown to its IR-fixed point $b_\mathrm{R} = b^*$, what implies that also the scaling exponent $\nu(b_\mathrm{R})$, defined by

$$\nu(b_\mathrm{R}) = \frac{2-D}{2} - \frac{1}{2}\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z \tag{20}$$

has flown to its IR-fixed point $\nu^*$. The $\beta$-function associated to the coupling $\eta$ is

$$\begin{aligned}\beta_\eta &= \mu\frac{\partial}{\partial\mu}\bigg|_0 \eta_\mathrm{R} \\ &= \eta_\mathrm{R}\left(-\delta + (1-d/2)\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z + \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_\lambda\right).\end{aligned} \tag{21}$$

Suppose now that $\eta$ has a nontrivial fixed point $\eta^*$ for $\eta_\mathrm{R} > 0$, *i.e.* $\beta_\eta(\eta^*, b^*) = 0$ and $\frac{\partial}{\partial\eta_\mathrm{R}}\beta_\eta(\eta_\mathrm{R}, b^*)|_{\eta_\mathrm{R}=\eta^*} > 0$. (We will show below that at leading order such a fixed point exists.) We now express $\mu\frac{\partial}{\partial\mu}\big|_0 \ln Z$ by its value at the IR-fixed point $b_\mathrm{R} = b^*$

$$2(\nu^* - \nu) = -\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z. \tag{22}$$

We can then solve for $\mu\frac{\partial}{\partial\mu}\big|_0 \ln Z_\lambda$

$$\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_\lambda = \delta + (2-d)(\nu^* - \nu). \tag{23}$$

The exponent $z$ is given by

$$z = (D + 2\nu^* - \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_\lambda)/\nu^*. \tag{24}$$

The last two equations can be combined to give

$$z = d. \tag{25}$$

This relation is valid as long as $\eta_\mathrm{R}$ has flown to a nontrivial fixed point $\eta^* > 0$. We will first study the stability of the fixed point $\eta_\mathrm{R} = 0$, before analyzing potential fixed points for $\eta_\mathrm{R} > 0$. Without selfavoidance this is simply the line with $\delta = 0$, see Figure 1. Selfavoidance however modifies this phase-separation line. To see this, look at $\beta_\eta$ at $\eta_\mathrm{R} \approx 0$:

$$\beta_\eta = (-\delta + (d-2)(\nu^* - \nu))\,\eta_\mathrm{R} + \eta_\mathrm{R}^2, \tag{26}$$

where we used the fact that $\mu\frac{\partial}{\partial\mu}\ln Z_\lambda = \mathcal{O}(\eta_\mathrm{R})$. The stability condition for the fixed point $\eta_\mathrm{R} = 0$ is therefore

$$\delta < (d-2)(\nu^* - \nu). \tag{27}$$

At 1-loop order, the separating line is

$$\delta = (d-2)\frac{\varepsilon}{8}\cdot \tag{28}$$

Numerical evaluation yields the thin line separating the regions with $z = d$ and $z = 2 + D/\nu^*$ in Figure 1. There is however *a priori* no reason to trust this estimate for membranes, *i.e.* $\varepsilon = 4$. We know however that in any dimension the Flory-estimate $\nu_\mathrm{Flory} = (2+D)/(2+d)$ is quite a good approximation for $\nu^*$ in the fractal phase, for polymers as well as for membranes [12,13]. Inserting this relation we obtain for the separatrix

$$d = 2(D+1). \tag{29}$$

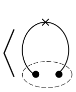
(This is the fat line between the regions with $z = d$ and $z = 2 + D/\nu^*$.) Let us stress that we only use the Flory-approximation to estimate $\nu^*$, but not any of the systematically wrong assumptions which have to be used to derive it.

Another possibility to get (29) is to demand that the value of $z$ is continuous on the phase separation line. The equivalence of the results obtained by the two methods is a consequence of the general structure of the renormalization group.

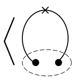
We also can give a rigorous bound for the phase separation line. As $\nu^* \leq 1$, hydrodynamics is always relevant for

$$d < \frac{8}{4-D}\cdot \tag{30}$$

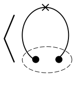
We still have to check that $\beta_\eta$ has a fixed point for $\eta_\mathrm{R} > 0$. At 1-loop order, $Z_\lambda$ is

$$Z_\lambda = 1 - \Big\langle$$

$$\Big|\bullet\Big\rangle_{\delta^{-1}}\frac{\eta_\mathrm{R}}{\delta} + \mathcal{O}(\eta_\mathrm{R}^2). \tag{31}$$

The diagram on the r.h.s. is the contraction of the hydrodynamic interaction only. Explicitly this is

$$\Big\langle$$

$$\Big|\bullet\Big\rangle_\mu \delta_{\alpha\beta} = \int_{x<\mu^{-1}}\int_k \left(\frac{\delta_{\alpha\beta}}{k^2} - \frac{k_\alpha k_\beta}{k^4}\right)\mathrm{e}^{-|x|^{2-D}k^2}$$
$$= \delta_{\alpha\beta}\frac{d-1}{2d(d-2)}\frac{1}{\delta}\mu^{-\delta}. \tag{32}$$

The residue is thus positive

$$\Big\langle$$

$$\Big|\bullet\Big\rangle_{\delta^{-1}} > 0. \tag{33}$$

This ensures the stability of the fixed point at least for small $\delta$.

In conclusion: We have shown that the dynamical field theory (9) for polymerized tethered membranes including hydrodynamics is renormalizable and that the dynamical scaling exponent $z$ is given to all orders in perturbation theory by $d$.

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